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Graph Intersection and Uniqueness Results for Some Nonlinear Elliptic Problems

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1. INTRODUCTION

This brief note concerns the uniqueness and graph intersection properties for positive solutions of a class of nonlinear Sturm–Liouville ordinary differential equations of the form

$$(\varphi u')' + g(x, u) f(u) = 0 \tag{1.1}$$

when $\varphi > 0$, f is continuously differentiable and concave, and g is non-increasing in u. No assumption is made about the sign of g so the nonlinearity may be both concave and convex in u on different parts of the x-domain. (Here prime denotes differentiation with respect to x, and u is a function of x.)

Such problems, when g takes both positive and negative values, arise in population genetics. (See [1, 5] for the biological background material.) Actually, the problems which arise in population genetics tend to be partial differential equations of the form

$$\begin{aligned} \Delta u + g(|\mathbf{x}|, u) f(u) &= 0 \quad \text{in} \quad \mathbf{R}^N, \\ u &> 0 \quad \text{in} \quad \mathbf{R}^N, \\ \lim_{|\mathbf{x}| \to \infty} u(\mathbf{x}) &= 0, \end{aligned}$$

(see [6, 7, 8]). However, Tertikas has shown in many cases that all

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solutions of this problem are in fact radially symmetric, in which case the problem reduces to an ordinary differential equation

$$(x^{N-1}u')' + x^{N-1}g(x, u)f(u) = 0 \qquad \text{on} \quad (0, \infty). \tag{1.2}$$

Our main observations concerning solutions of (1.2) follow as elementary corollaries of Theorem 2.1, which is the main result to be presented here. Basically what it says is that, provided hypothesis A of the next Section holds (and this is no more than a precise description of what has already been said: f concave, $g(x, \cdot)$ nonincreasing and $\varphi > 0$) then any two positive solutions of (1.1) are related by an integro-differential identity (2.6). An immediate consequence is that distinct positive solutions intersect at most once (Corollary 2.2). Other consequences are also immediate. For example, if $1/f$ is integrable in a neighbourhood of a, then distinct positive solutions u and v of (1.1) which converge to a as $x \to \infty$ cannot intersect on $(0, \infty)$, and if both satisfy the boundary condition at zero which says that $\varphi(x)\, u'(x) \to 0$ as $x \to 0$, then they must coincide (i.e., there is at most one positive solution). We also remark here that Theorem 2.1 gives uniqueness results for various nonlinear Dirichlet and Neumann boundary value problems on radially symmetric domains which may or may not be bounded. We also note that it applies equally to bounded and unbounded solutions on their intervals of existence.

This work arose from an investigation of problems first treated from the viewpoint of uniqueness theory by different and much more elaborate methods by Tertikas [7]. Uniqueness for problems of this type are well known when the nonlinearity $f(u)\, g(x, u)$ is concave in u (see, for example, [2]).

Other more subtle uniqueness theorems have been found by Serrin, Peletier, and McLeod (see [3, 4], and the references therein) under the assumption that the equation has no explicit dependence on x, which is clearly more restrictive than ours. In the case of (one-dimensional) nonlinear Sturm–Liouville problems some elementary uniqueness results, not contained herein, are obtained by other means in [9].

2. Graph Separation and Uniqueness for Odes

The following properties of functions f, g, and φ define a class of ordinary differential equations whose solutions enjoy certain important graph separation properties.

A: $\alpha > 0$ *and* $f\colon (0, \alpha) \to (0, \infty)$ *is a continuously differentiable concave function*; $g\colon (a, b) \times (0, \alpha) \to \mathbf{R}$ *is continuous and* $g(x, \cdot)$ *is nonincreasing for* *each* $x \in (a, b)$; $\varphi\colon (a, b) \to (0, \infty)$ *is differentiable and* fg *is locally Lipschitz in the sense that for each* $(x, u) \in (a, b) \times (0, \alpha)$ *there exists* $K = K(x, u)$ *and* $\delta = \delta(x, u)$ *such that*

$$|f(v)\, g(y, v) - f(w)\, g(y, w)| \leqslant K\, |v - w|$$

when

$$|x - y| + |u - v| + |u - w| < \delta.$$

Remark. The sign of g is not specified by A.

Our main observation is the following. Here prime denotes differentiation.

THEOREM 2.1. *Suppose that* A *holds and that* u *and* v *are solutions of*

$$(\varphi(x)\, w'(x))' + g(x, w(x)) f(w(x)) = 0, \qquad x \in (a, b), \tag{2.1}$$

$$w(x) \in (0, \alpha),\ x \in (a, b). \tag{2.2}$$

For $x, \delta \in (a, b)$ *put*

$$r_\delta(x) = \varphi(x) \left[\exp\left\{ \int_\delta^x f'(v(t)) \left(\frac{u'(t)}{f(u(t))} + \frac{v'(t)}{f(v(t))} \right) dt \right\} \right] \left[\frac{u'(x)}{f(u(x))} - \frac{v'(x)}{f(v(x))} \right]. \tag{2.3}$$

(i) *If* $u \geqslant v$ *on* (a, b), *then* r_δ *is finite and nondecreasing on* (a, b) *for all* $\delta \in (a, b)$.

(ii) *For* δ_1 *and* $\delta_2 \in (a, b)$ *there exists a constant* $k = k(\delta_1, \delta_2) > 0$ *such that*

$$r_{\delta_1}(x) = k r_{\delta_2}(x), \qquad x \in (a, b).$$

(iii) *If* $u' \neq 0$, $f'(u) > 0$, *and* $u \geqslant v$ *on* (a, b), *then*

$$r_\delta(x) = \varphi(x) \left(\frac{f(u(x))}{f(u(\delta))} \right)^\beta \left(\frac{f(v(x))}{f(v(\delta))} \right) \left(\frac{u'(x)}{f(u(x))} - \frac{v'(x)}{f(v(x))} \right), \tag{2.4}$$

where $\beta = \beta(x, \delta)$ *is given by*

$$\beta = \operatorname{sgn}(x - \delta) f'(v(\xi))/f'(u(\xi)), \qquad \text{and} \qquad |\beta| \geqslant 1,$$

for some ξ *between* δ *and* x (ξ *depends on* δ *and* x).

(iv) *If $u \geqslant v$ and $u' \leqslant 0$ on (δ, b), then*

$$|r_\delta(x)| \leqslant \frac{|\varphi(x)\, u'(x) f(v(x))| + |\varphi(x)\, v'(x) f(u(x))|}{f(u(\delta)) f(v(\delta))}, \qquad x \in (\delta, b). \tag{2.5}$$

If $u' \geqslant 0$ and $u \geqslant v$ on (a, δ) then (2.5) *holds on* (a, δ).

Remark. This result is independent of the Lipschitz continuity assumption in A.

Proof. Let

$$W(x) = \varphi(x) \left\{ \frac{u'(x)}{f(u(x))} - \frac{v'(x)}{f(v(x))} \right\}, \qquad x \in (a, b).$$

A calculation using the fact that u and v satisfy (2.1) gives that

$$\begin{aligned} &W'(x) + f'(v(x)) \left\{ \frac{u'(x)}{f(u(x))} + \frac{v'(x)}{f(v(x))} \right\} W(x) \\ &\quad = g(x, v(x)) - g(x, u(x)) + \varphi(x)(f'(v(x)) - f'(u(x)))\{u'(x)/f(u(x))\}^2 \\ &\quad \geqslant 0 \qquad \text{on} \quad (a, b) \text{ if } u \geqslant v, \text{ since A holds.} \end{aligned} \tag{2.6}$$

For any $\delta \in (a, b)$, $v(\delta) \in (0, \alpha)$ by (2.2), and so by A

$$H_\delta(x) = \exp \left\{ \int_\delta^x f'(v(t)) \left(\frac{u'(t)}{f(u(t))} + \frac{v'(t)}{f(v(t))} \right) dt \right\}$$

is a positive integrating factor which is finite on (a, b). Hence, $W(x)\, H_\delta(x)$ is nondecreasing on (a, b) as required in (i). Clearly,

$$\begin{aligned} H_{\delta_1}(x) &= \exp \left\{ \int_{\delta_1}^{\delta_2} f'(v(t)) \left(\frac{u'(t)}{f(u(t))} + \frac{v'(t)}{f(v(t))} \right) dt \right\} H_{\delta_2}(x) \\ &= k H_{\delta_2}(x). \end{aligned} \tag{2.7}$$

This gives (ii). Now

$$\begin{aligned} H_\delta(x) &= \left(\frac{f(v(x))}{f(v(\delta))} \right) \exp \left\{ \frac{\displaystyle\int_\delta^x \frac{f'(v(t))\, u'(t)}{f(u(t))}\, dt}{\displaystyle\int_\delta^x \frac{f'(u(t))\, u'(t)}{f(u(t))}\, dt} \log \left(\frac{f(u(x))}{f(u(\delta))} \right) \right\} \\ &= \left(\frac{f(v(x))}{f(v(\delta))} \right) \exp \left\{ \frac{f'(v(\xi))}{f'(u(\xi))} \log \left(\frac{f(u(x))}{f(u(\delta))} \right) \right\}, \end{aligned}$$

if $f'(u(x))\,u'(x) \neq 0$, $x \in (\delta, b)$, by Cauchy's version of the mean value theorem. Now $\beta \geqslant 1$ since $u \geqslant v$, f is concave and $f'(u) > 0$. Similarly if $x < \delta$. This yields (iii).

If $u' \leqslant 0$ and $u \geqslant v$ on (δ, b), then $f'(v(t))/f(u(t)) \geqslant f'(u(t))/f(u(t))$ implies that

$$\frac{f'(v(t))\,u'(t)}{f(u(t))} \leqslant \frac{f'(u(t))\,u'(t)}{f(u(t))},$$

from which (2.5) follows after substituting this inequality in the definition of H_δ. This completes the proof. Q.E.D.

COROLLARY 2.2. *Suppose that* A *holds, that* u *and* v *satisfy* (2.1) *and* (2.2), *and that* $u(\delta) = v(\delta)$ *for some* $\delta \in (a, b)$.

(a) *Then either* $u(x) = v(x)$ *for all* $x \in (a, b)$, *or* $u(x) \neq v(x)$ *for all* $x \in (a, b)\backslash\{\delta\}$.

(b) *If* $u \not\equiv v$, *suppose without loss that* $u(x) > v(x)$ *on* (a, δ). *Then*

$$r_\delta(x) < r_\delta(\delta) < 0, \qquad x \in (a, \delta).$$

Proof. Suppose that $u \not\equiv v$ on (a, b). Then from the Lipschitz condition in A we conclude that $u'(\delta) \neq v'(\delta)$. Without loss, suppose $u'(\delta) > v'(\delta)$. Then $r_\delta(\delta) > 0$ and hence $r_\delta(x) > 0$ on $[\delta, c]$ if $u > v$ on (δ, c). If c is the first zero of $u - v$ to the right of δ, then $r_\delta(c) \leqslant 0$ which is a contradiction. Hence there is only one zero of $u - v$ in (a, b).

Part (b) is immediate in that case since $r_\delta(\delta) < 0$ and the theorem holds. Q.E.D.

As a consequence we obtain our first nonintersection result.

THEOREM 2.3. *Suppose that* A *holds with* $(a, b) = (0, \infty)$ *and that* u *and* v *are distinct solutions of*

$$(\varphi(x)\,w'(x))' + g(x, w(x))\,f(w(x)) = 0,$$
$$w(x) \in (0, \alpha),\ x \in (0, \infty).$$

Then either

$$u(x) \neq v(x) \qquad \textit{for all} \quad x \in (0, \infty),$$

or

$$\lim_{x \to \infty} \int_{v(x)}^{u(x)} \frac{dw}{f(w)} \textit{ exists and is nonzero.}$$

This limit may be infinite.

Proof. If u and v are distinct, then it follows from Corollary 2.2(a) that $u(x) \neq v(x)$ for all x sufficiently large. Without loss of generality suppose that $u(x) > v(x)$ for all $x > X$. Let $X \in [0, \infty)$ be the smallest such X. If $X > 0$, then $u(X) = v(X)$, $u'(X) > v'(X)$ and so $r_X(x) \geqslant r_X(X) > 0$ for all $x \in [X, \infty)$. Now it follows that

$$\frac{u'(x)}{f(u(x))} - \frac{v'(x)}{f(v(x))} > 0, \qquad x \in [X, \infty),$$

and so

$$\int_{v(x)}^{u(x)} \frac{dw}{f(w)} = \int_X^x \left(\frac{u'(t)}{f(u(t))} - \frac{v'(t)}{f(v(t))} \right) dt > 0.$$

The result follows by letting x tend to ∞. Q.E.D.

THEOREM 2.4. *Suppose that* A *holds on* $(0, \infty)$, $a \in [0, \alpha]$ *and* $1/f$ *is integrable in a neighborhood of* a *in* $(0, \alpha)$. *Then there exists at most one function* u *satisfying the boundary-value problem*

$$(\varphi(x)\, u'(x))' + g(x, u(x)) f(u(x)) = 0, \qquad x \in (0, \infty),$$

$$\lim_{x \to 0} u(x) \in (0, \alpha),$$

$$u'(x) \quad \text{is bounded as } x \to 0,$$

$$\varphi(x)\, u'(x) \to 0 \quad \text{as} \quad x \to 0,$$

$$u(x) \to a \quad \text{as} \quad x \to \infty.$$

Proof. Suppose u and v both satisfy this boundary-value problem. Then since $u(x)$, $v(x) \to a$ as $x \to \infty$, and $1/f$ is integrable in a neighbourhood of a in $(0, \alpha)$ we conclude from Theorem 2.3 that $u(x) \neq v(x)$, $x \in (0, \infty)$. Without loss of generality suppose that $u(x) > v(x)$, $x \in (0, \infty)$. The boundary conditions on u and v at zero and the concavity of f in (2.3) mean that $r_1(x) \to 0$ as $x \to 0$. Hence, $r_1(x) \geqslant 0$ on $(0, \infty)$, and so

$$\int_{v(x)}^{u(x)} \frac{du}{f(u)} = \int_x^\infty \left(\frac{v'}{f(v)} - \frac{u'}{f(u)} \right) dt \leqslant 0,$$

which contradicts the fact that $u > v$ on $(0, \infty)$. This contradiction establishes the theorem. Q.E.D.

Remark. The hypotheses of the preceding theorem admit the case when the equation is linear, i.e., when $g(x, u) f(u) = \alpha(x) u$, $\alpha \neq 0$. However, since

f is concave $(u/f(u))$ is nondecreasing, and since $g(x, u) = \alpha(x)(u/f(u))$ is nonincreasing we conclude that $\alpha \leqslant 0$ everywhere. In this case, if $g(x, u) = \alpha(x)\, u^{1/2}(x)$ and $f(u) = u^{1/2}$, then hypothesis A is satisfied. To obtain further uniqueness we impose a one-sided integrability condition on g at infinity.

B: *$g(x, 0) \leqslant h(x)$ for all $x \geqslant Y$ for some $Y > 0$, where $h \geqslant 0$ is integrable on* (Y, ∞).

LEMMA 2.5. *Suppose that* A *and* B *hold, that*

$$(\varphi(x)\, u'(x))' + g(x, u(x)) f(u(x)) = 0, \qquad x \in (Y, \infty),$$

$$f(u(x)) \text{ is bounded on } (Y, \infty)$$

and

$$\liminf_{x \to \infty} \varphi(x)\, u'(x) < \infty.$$

Then $\lim_{x \to \infty} \varphi(x)\, u'(x)$ *exists and is finite.*

Proof. Let $x_n \to \infty$ and $M \in \mathbf{R}$ be such that $\varphi(x_n)\, u'(x_n) \leqslant M$ for all n. Then for all $x > Y$ sufficiently large and $x_n > x$ we have

$$\begin{aligned}
\varphi(x)\, u'(x) &= \varphi(x_n)\, u'(x_n) + \int_x^{x_n} g(t, u(t)) f(u(t))\, dt \\
&\leqslant M + \int_x^{x_n} g(t, 0) f(u(t))\, dt, \qquad \text{since } f(u(t)) \geqslant 0, \text{ by A}, \\
&\leqslant M + \int_x^{x_n} h(t) f(u(t))\, dt \\
&\leqslant M + \int_Y^{\infty} h(t) f(u(t))\, dt < \infty.
\end{aligned}$$

Therefore, $\varphi(x)\, u'(x) \leqslant K$, say, for all x sufficiently large. Now for x sufficiently large

$$\begin{aligned}
0 &= (\varphi(x)\, u'(x))' + g(x, u(x)) f(u(x)) \\
&\leqslant (\varphi(x)\, u'(x))' + g(x, 0) f(u(x)) \\
&\leqslant (\varphi(x)\, u'(x))' + ch(x) \qquad (\text{where } c > 0 \text{ is a constant}) \\
&= \left\{ \varphi(x)\, u'(x) + c \int_Y^x h(t)\, dt \right\}'.
\end{aligned}$$

Moreover $\varphi(x)\, u'(x) + c \int_Y^x h(t)\, dt \leqslant K + c \int_Y^\infty h(t)\, dt < \infty$, and hence $\lim_{x \to \infty} \{\varphi(x)\, u'(x) + c \int_Y^x h(t)\, dt\}$ exists. But $c \int_Y^x h(t)\, dt \to c \int_Y^\infty h(t)\, dt$, and so the required result is proved. Q.E.D.

Remark. An extra assumption, such as B, is often necessary in order to obtain existence results for this class of equations. It is useful to note that if $g(x, 0) \leqslant 0$, $x \geqslant Y$, then the hypotheses of Theorem 2.5 hold *and* the solutions of (2.1) are monotone on (Y, ∞). This is important in the light of the hypotheses of the next result.

THEOREM 2.6. *Suppose that* A *and* B *hold on* $(0, \infty)$, $f(0) = 0$, *and* u *and* v *are two solutions of the problem*

$$(\varphi(x)\, w'(x))' + g(x, w(x))\, f(w(x)) = 0, \qquad x \in (0, \infty),$$

$$w(x) \in (0, \alpha),\ x \in (0, \infty),$$

$$\lim_{x \to 0} w(x) \in (0, \alpha), \qquad \lim_{x \to \infty} w(x) = 0,$$

with $w' \leqslant 0$ *in a neighbourhood of* $+\infty$, $w'(x)$ *bounded in a neighbourhood of* 0, *and*

$$\varphi(x)\, w'(x) \to 0 \qquad as \quad x \to 0.$$

Then

$$u'(x)^2 \{f'(u(x)) - f'(v(x))\} = 0, \qquad x \in (0, \infty),$$

$$u'(x) f(v(x)) - v'(x) f(u(x)) = 0, \qquad x \in (0, \infty),$$

and

$$g(x, u(x)) = g(x, v(x)), \qquad x \in (0, \infty).$$

Consequently if f' *is not a constant in any neighbourhood of* 0, *or if* $g(x, \cdot)$ *is not constant in any neighbourhood of* 0 *for* x *large, then* $u = v$.

Proof. Suppose that u and v are distinct solutions. Let $\delta \in (0, \infty)$ be chosen so that $u(x) > v(x)$ on $(0, \delta)$. This can be done because of Corollary 2.2. Then because of the boundary condition at $x = 0$, we know that $r_\delta(x) \to 0$ as $x \to 0$, and so $u(x) > v(x)$ and $r_\delta(x) \geqslant 0$, $x \in (0, \infty)$. Now $\liminf_{x \to \infty} \varphi(x)\, u'(x) \leqslant 0$, and hence by Lemma 2.6, $\varphi(x)\, u'(x)$ has a finite limit as $x \to \infty$. Similarly for $\varphi(x)\, v'(x)$.

We choose δ_1 such that $u' \leqslant 0$ on (δ_1, ∞). Then $r_\delta = k r_{\delta_1}$, and $r_{\delta_1}(x) \to 0$ as $x \to \infty$ by (2.5). Hence, $r_\delta(x) = 0$ on $(0, \infty)$, and the result now follows from the equation for W in the proof of Theorem 2.1. Q.E.D.

In case f is linear, i.e., $f(u) = \lambda u$, we have the following consequence.

COROLLARY 2.7. *Suppose that* $\lambda > 0$, A *and* B *hold and* u, v *are two solutions of the problem*

$$(\varphi(x)\, w'(x))' + \lambda g(x, w(x))\, w(x) = 0, \qquad x \in (0, \infty),$$

$$w(x) \in (0, \alpha), \qquad x \in (0, \infty),$$

$$\lim_{x \to 0} w(x) \in (0, \alpha), \qquad \lim_{x \to \infty} w(x) = 0,$$

with $w' \leqslant 0$ *in a neighbourhood of* ∞, $w'(x)$ *bounded in a neighbourhood of* 0 *and*

$$\varphi(x)\, w'(x) \to 0 \qquad as \quad x \to 0.$$

Then

$$u(x) = cv(x), \qquad x \in (0, \infty)$$

for some suitable $c > 0$ *and*

$$g(x, u(x)) = g(x, v(x)), \qquad x \in (0, \infty).$$

Proof. From Theorem 2.6 we conclude that

$$u'(x)\, v(x) - v'(x)\, u(x) = 0, \qquad x \in (0, \infty).$$

Hence,

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = 0, \qquad x \in (0, \infty),$$

or

$$u(x) = cv(x), \qquad x \in (0, \infty). \qquad \text{Q.E.D.}$$

Remark. The previous corollary gives uniqueness (up to normalisation) of positive eigenfunctions of the problem in case $g(x, u) = g(x)$.

Remark. It is important to observe that in the proof of the preceding theorem it is the function which is larger at infinity which is required to be monotone. In fact there is no monotonicity assumption at all on the other function v; it suffices merely to know that $\varphi v'$ is bounded at ∞.

Here is another application which yields uniqueness without monotonicity assumptions on u.

THEOREM 2.8. *Suppose that* A *and* B *hold on* $(0, \infty)$, $f' \leqslant 0$ *on* (γ, α) *for some* $\gamma \in (0, \alpha)$, f *is continuous on* $(0, \alpha]$ *and* $f(\alpha) = 0$. *Suppose that* u *and* v *are distinct solutions of*

$$(\varphi(x)\, w'(x))' + g(x, w(x)) f(w(x)) = 0,$$

$$w(x) \in (0, \alpha),\ x \in (0, \infty),$$

$$w(x) \to \alpha \qquad \textit{as} \quad x \to \infty,$$

$$\liminf_{x \to \infty} \varphi(x)\, w'(x) < \infty,$$

$$\lim_{x \to 0} w(x) \in (0, \alpha),$$

$$\varphi(x)\, w'(x) \to 0 \qquad \textit{as} \quad x \to 0,$$

$w'(x)$ *is bounded in a neighbourhood of* 0.

Then (without loss of generality) $u(x) > v(x)$ *in a neighbourhood of* ∞ *and either*

$$\liminf_{x \to \infty} \frac{\{f(v(x))\}^2}{f(u(x))} > 0,$$

or

$$g(x, u(x)) - g(x, v(x)) = u'(x)^2 \{f'(u(x)) - f'(v(x))\} = 0, \qquad x \in (0, \infty),$$

and

$$u'(x) f(v(x)) - v'(x) f(u(x)) = 0, \qquad x \in (0, \infty).$$

Proof. Since u and v have graphs which intersect at most once we may suppose that $u > v$ on (δ, ∞) for some $\delta > 0$. Let δ be chosen so that, in addition, $f'(v(x)) \leqslant 0$ for all $x \in (\delta, \infty)$. If the graphs of u and v intersected at $X < \delta$, then $r_\delta(X) > 0$, whence $r_\delta(x) > 0$ on (δ, ∞) by Theorem 2.1. If the graphs do not intersect, then the boundary conditions at 0 ensure that $r_\delta(0)=0$ and $u>v$ on $(0, \infty)$. Again by Theorem 2.1, $r_\delta(x)\geqslant 0$ on (δ, ∞). Hence,

$$f'(v(x))\leqslant 0 \qquad \text{and} \qquad \frac{u'(x)}{f(u(x))}-\frac{v'(x)}{f(v(x))}\geqslant 0, \qquad x\in(\delta, \infty),$$

which yields

$$f'(v(x))\frac{u'(x)}{f(u(x))}\leqslant f'(v(x))\frac{v'(x)}{f(v(x))}, \qquad x\in(\delta, \infty).$$

Taking these inequalities into account, the expression (2.3) for $r_\delta(x)$ yields the estimate

$$0\leqslant r_\delta(x)\leqslant \varphi(x)\left(\frac{f(v(x))}{f(v(\delta))}\right)^2\left(\frac{u'(x)}{f(u(x))}-\frac{v'(x)}{f(v(x))}\right).$$

But by our hypotheses and Lemma 2.5, this yields that

$$0\leqslant \liminf_{x\to\infty} r_\delta(x)\leqslant (\text{const})\liminf_{x\to\infty}\frac{f(v(x))^2}{f(u(x))}.$$

If the right-hand side is zero then the result follows from the equation for W in the proof of Theorem 2.1. Q.E.D.

The next result is typical of what can be obtained on bounded domains.

THEOREM 2.9. *Suppose that* A *and* B *hold on* $(0, R)$, φ *is bounded on* $(0, R]$, $f(0)=0$, *and* u *and* v *are two solutions of the problem*

$$(\varphi(x)\,w'(x))'+g(x, w(x))f(w(x))=0, \qquad x\in(0, R]$$

w' is bounded on $(0, R)$ and $w'\leqslant 0$ in a neighbourhood of R,

$$w(x)\in(0, \alpha),\ x\in(0, \infty),$$

$$\lim_{x\to 0} w(x)\in(0, \alpha), \qquad \lim_{x\to 0}\varphi(x)\,w'(x)=0,$$

$$w(R)=0.$$

Then the conclusion of the preceding theorem holds on $(0, R)$.

Proof. The proof is identical to that of Theorem 2.6. Q.E.D.

3. Uniqueness for Semi-linear Elliptic Problems

The preceding sections contain results whose significance for nonlinear Sturm–Liouville and semi-linear elliptic problems will now be explained briefly. We begin with the one-dimensional case.

3.1. *Uniqueness for Sturm–Liouville Problems*

Consider the problem

$$u''(x) + g(x, u(x))f(u(x)) = 0, \qquad x \in \mathbf{R}, \tag{3.1}$$

$$u(x) \text{ is bounded on } \mathbf{R}, \tag{3.2}$$

$$u(x) \in (0, \alpha),\ x \in \mathbf{R}, \tag{3.3}$$

$$xu'(x) \leqslant 0 \qquad \text{for} \quad |x| \text{ sufficiently large.} \tag{3.4}$$

Suppose that A and B hold on $\mathbf{R}$ with $\varphi(x) \equiv 1$.

THEOREM 3.1. *Suppose that u and v are both solutions of* (3.1)–(3.4). *Then their graphs do not intersect. Consequently we may suppose that $u(x) > v(x)$, $x \in \mathbf{R}$, and then*

$$g(x, u(x)) = g(x, v(x)), \qquad x \in \mathbf{R},$$

$$u'(x)f(v(x)) - v'(x)f(u(x)) = 0, \qquad x \in \mathbf{R},$$

and

$$u'(x)^2 \{f'(v(x)) - f'(u(x))\} = 0, \qquad x \in \mathbf{R},$$

from which uniqueness may be inferred depending upon the behaviour of f and g.

Proof. Since u is bounded on $\mathbf{R}$, $\liminf_{x \to \infty} u'(x) \leqslant 0$, and so $u'(x)$ converges to a limit as $|x| \to \infty$, by Lemma 2.5 (since $\varphi = 1$). Clearly this limit is zero. Since the graphs of u and v intersect at most once, we may suppose that $u(x) > v(x)$ for x sufficiently large. But $r_\delta(x) \to 0$ as $x \to \infty$ by (2.5), and we conclude that the graphs of u and v do not intersect on $\mathbf{R}$. Moreover, $r_\delta(x) \to 0$ as $x \to -\infty$, and so $r_\delta(x)$ is constant on $\mathbf{R}$. This completes the proof of the theorem. Q.E.D.

Related results based on Theorem 2.8 can also be proved, but we will not labour this point any further.

3.2. *Semi-Linear Elliptic Problems*

We consider the problem of radially symmetric solutions in $\mathbf{R}^N$, or in a ball in $\mathbf{R}^N$, of the form

$$\Delta u(\mathbf{x}) + g(|\mathbf{x}|, u(\mathbf{x}))f(u(\mathbf{x})) = 0.$$

If $x = |\mathbf{x}|$ and $u(\mathbf{x}) = w(x)$, then the equation for w is familiar, namely

$$(x^{N-1}w'(x))' + x^{N-1}g(x, w(x))f(w(x)) = 0. \tag{3.5}$$

Let us suppose throughout that A holds for $x \in (0, \infty)$ with $\alpha = +\infty$ for convenience. We will only consider nonconstant classical radially symmetric solutions here; thus in particular

$$f(w(0)) \in (0, \infty) \qquad \text{and} \qquad w'(0) = 0. \tag{3.6}$$

(However, it should be noted that the results of the preceding section can apply to solutions with a singularity at the origin, in certain circumstances.)

The first result concerns Dirichlet and Neumann problems on a ball in $\mathbf{R}^N$, and is based on Theorem 2.8, for any $N \in \mathbf{N}$.

THEOREM 3.2. *Suppose that* A *holds, f' is continuous on* $[0, \infty)$, $f(0) = 0$, *and that u and v are distinct classical solutions of* (3.5) *and* (3.6) *on* $[0, R]$ *which are positive on* $[0, R)$ *and which satisfy either*

$$u(R) = v(R) = 0 \tag{3.7}$$

or

$$u'(R) = v'(R) = 0. \tag{3.8}$$

If u' and v' are nonpositive in a neighbourhood of R, then

$$u(x) \neq v(x), \qquad x \in (0, R).$$

If $u(x) > v(x)$, *then*

$$g(x, u(x)) = g(x, v(x)),$$
$$u'(x)^2 \{f'(v(x)) - f'(u(x))\} = 0, \qquad x \in (0, R],$$

and

$$u'(x)f(v(x)) - v'(x)f(u(x)) = 0, \qquad x \in (0, R).$$

Proof. We may suppose that $u(x) > v(x)$ for all x close to 0. Let $\delta \in (0, R)$ be such that $u(x) > v(x)$ for all $x \in (0, \delta)$. Now because of the boundary condition at $x = 0$, $r_\delta(0) = 0$ and by (2.5) we find that

$$r_\delta(x) \to 0 \qquad \text{as} \quad x \to R \qquad \text{if} \quad (3.7) \text{ holds}.$$

If (3.8) holds, then

$$\lim_{x \to R} r_\delta(x) = 0.$$

In both cases we conclude that $u(x) \neq v(x)$, $x \in (0, R)$, and hence that $r_\delta(x)$ is constant on $(0, R)$. An appeal to the differential equation for W in the proof of Theorem 2.1 now yields the required result. Q.E.D.

We finish this brief section by illustrating our results for semi-linear problems on $\mathbf{R}^N$, $N = 2$ and $N \geqslant 3$.

THEOREM 3.3. *Suppose that $N = 2$ and that* A *and* B *hold and* $f(0) = 0$. *Let u and v be two solutions of* (3.5) *and* (3.6) *on* $(0, \infty)$ *which are positive on* $[0, \infty)$ *with*

$$u(x) \text{ and } v(x) \text{ monotone nonincreasing} \qquad \text{as} \quad x \to \infty$$

$$f'(u(x))\, u'(x) < 0 \text{ and } f'(v(x))\, v'(x) < 0 \qquad \text{as} \quad x \to \infty.$$

Then the conclusion of the preceding theorem holds on $(0, \infty)$.

Proof. Since u and v are nonincreasing $x \to \infty$, we find that $\liminf_{x \to \infty} xu'(x) \leqslant 0$, and similarly for v. Hence, $xu'(x)$ and $xv'(x)$ have finite limits as $x \to \infty$ ($\lim_{x \to \infty} xu'(x) \leqslant 0$). The proof based on Theorem 2.1 is now familiar. Q.E.D.

THEOREM 3.4. *Suppose that $N \geqslant 3$ and that* A *and* B *hold with* $f(0) = 0$. *Let u and v be two solutions of* (3.5) *and* (3.6) *on* $(0, \infty)$ *which are positive on* $[0, \infty)$ *with*

$$u(x) \text{ and } v(x) \text{ monotone nonincreasing} \qquad \text{as} \quad x \to \infty$$

$$f'(u(x))\, u'(x) < 0 \text{ and } f'(v(x))\, v'(x) < 0 \qquad \text{as} \quad x \to \infty$$

and

$$u(x) \to 0, \qquad v(x) \to 0 \qquad \text{as} \quad x \to \infty.$$

Then the conclusion of Theorem 3.2 *holds on* $(0, \infty)$.

Proof. Since u' and v' are nonpositive at infinity by hypothesis we find that $\lim_{x \to \infty} x^{N-1} u'(x)$ is finite. Similarly for v. The proof based on Theorem 2.1 is by now familiar. Q.E.D.

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